Eur. Phys. J. C 51, 75–82 (2007)
DOI 10.1140/epjc/s10052-007-0292-9

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article – Theoretical Physics

A renormalization group analysis of leading logarithms in ChPT

M. Bissegger[a], A. Fuhrer[b]

Institute for Theoretical Physics, University of Bern, Sidlerstr. 5, 3012 Bern, Switzerland

Received: 17 January 2007 /
Published online: 17 April 2007 –

Abstract. We give strong evidence that the linear sigma model at small external momenta is an effective theory for the leading logarithms of chiral perturbation theory. Based on this evidence an attempt is made to sum the leading logarithms of chiral perturbation theory to all orders. We illustrate why this summation nonetheless fails when one uses standard renormalization group techniques of renormalizable quantum field theories.

1 Introduction

The theory of the strong interaction, quantum chromodynamics, does not allow for the application of the powerful perturbation techniques in the low-energy region. It is, however, possible to use the symmetries of QCD to construct an effective theory in the low-energy region, chiral perturbation theory (ChPT) [1–3], which allows for a systematic perturbative expansion of Green functions in powers of external momenta and quark masses.

In actual calculations in ChPT one expects the dominant contribution to stem from the leading effective Lagrangian, which generates the leading chiral logarithm. Even if the latter do not always dominate, it would be very interesting to know the leading chiral logarithms to every order in the perturbative expansion, and to sum them up.

In a recent publication [4], we presented a procedure which allows the calculation of leading logarithms of certain Green functions in the chiral limit rather easily. In the present article, we address the question whether it is possible to sum up these leading logarithms to all orders.

In a given renormalizable quantum field theory, resummation of logarithms is based on the renormalization group equations (RGE). However, chiral perturbation theory is not renormalizable, and the structure of the RGE is therefore more involved [5]. In order to avoid the problems introduced by the nonrenormalizable nature of chiral perturbation theory, we propose to consider a theory that *is renormalizable* and reproduces the leading logarithms of chiral perturbation theory. It is then natural to expect that the summation of logarithms in this renormalizable theory can be performed by the use of the RGE.

It has been known since long that the tree-level graphs of the linear sigma model reproduce, at small momenta, the results of current algebra. In modern language, this means that they agree with the tree-level graphs of ChPT. It was shown in [2] that this persists at one-loop order: provided that the low-energy couplings in the chiral Lagrangian are properly adapted, the Green functions evaluated in the linear sigma model at one-loop order agree with the result of ChPT at order p^4; see also [6]. This shows that the linear sigma model is a promising candidate for a renormalizable theory that generates the leading logarithms in ChPT.

To carry the comparison between ChPT and the linear sigma model to higher orders in the momentum expansion, we consider the correlator of two scalar quark currents,

$$H(s) = \mathrm{i}\int \mathrm{d}x e^{\mathrm{i}px}\langle 0|TS^0(x)S^0(0)|0\rangle\,,$$
$$S^0 = \bar{u}u + \bar{d}d;\quad s = p^2\,, \tag{1}$$

in a world with two flavors in the chiral limit $m_u = m_d = 0$. Its leading chiral logarithms have been worked out in ChPT to five-loop accuracy in [4]. This article is devoted to an analysis of this correlator in the framework of the linear sigma model, addressing the questions just raised: does the linear sigma model reproduce these logarithms, and if so, can they be summed?

The structure of the article is as follows: In Sect. 2, we recall the structure of the leading logarithms of $H(s)$ in chiral perturbation theory in the chiral limit. In Sect. 3, we calculate the leading logarithms of the scalar two-point function – which corresponds to the quantity $H(s)$ – in the linear sigma model, and we show in Sect. 4 that this theory reproduces the leading logarithms of chiral perturbation theory up to and including two loops in this case. For this reason, we believe that the linear sigma model indeed is a renormalizable effective theory suitable to calculate the leading logarithms in ChPT. In Sect. 5, we consider the summation of leading logarithmic singularities in both the symmetric and the spontaneously broken phase of the linear sigma model. In the following section, Sect. 6, we apply this technique to the scalar two-

[a] e-mail: bissegg@itp.unibe.ch
[b] e-mail: afuhrer@itp.unibe.ch

point function in the spontaneously broken phase. We are able to sum up a certain class of logarithmic terms, and explain why an explicit summation of all leading logarithms is not possible with this technique. Finally, Sect. 7 contains a summary and concluding remarks. The appendices contain several technical aspects of our investigation: in Appendix A, we present expressions for the relevant triangle graphs, whereas the two-loop diagrams needed in the calculation of the two-point function are displayed and discussed in Appendix B. A dispersive calculation, used as a check on certain two-loop diagrams, is presented in Appendix C, and scale dependent logarithms are summed up in Appendix D.

2 Leading logarithms in ChPT

In the chiral limit, the low-energy expansion of the scalar correlator can be written as

$$H(s) = \frac{B^2}{16\pi^2}\{P_0(s,\bar\mu)+P_1(s,\bar\mu)L+P_2(s,\bar\mu)L^2+\cdots\}\,,$$
$$L = \ln\left(-\frac{s}{\bar\mu^2}\right)\,, \tag{2}$$

where P_i are polynomials in $N = s/(16\pi^2F^2)$. The quantities B, F are the two low-energy constants (LECs) at leading order in the chiral expansion [2], and the running scale of ChPT is denoted by $\bar\mu$. The *leading terms* $\bar P_i$ of the polynomials P_i – which are the coefficients of the leading logarithms – are known up to five loops [4],

$$\bar P_0 = 0\,,\qquad \bar P_1 = -6,\qquad \bar P_2 = 6N\,,$$
$$\bar P_3 = -\frac{61}{9}N^2,\quad \bar P_4 = \frac{68}{9}N^3\,,\quad \bar P_5 = -\frac{140\,347}{16\,200}N^4\,. \tag{3}$$

The full polynomials P_i differ from $\bar P_i$ by terms of order s^i and higher.

3 Chiral logarithms in the linear sigma model

We first introduce our notation of the linear sigma model and work out the quantity in the linear sigma model that corresponds to the scalar correlator $H(s)$. Then we calculate the two-loop leading logarithm of this quantity in the linear sigma model.

3.1 Notation

The Lagrangian of the $O(4)$ linear sigma model [7, 8] coupled to external scalar sources reads

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi^a\partial^\mu\varphi^a + \frac{m^2}{2}\varphi^a\varphi^a - \frac{g}{4}(\varphi^a\varphi^a)^2 + j^a\varphi^a\,,$$
$$a = 0,\ldots,3\,. \tag{4}$$

If $m^2 > 0$, the $O(4)$ symmetry is spontaneously broken down to $O(3)$, leading to three Goldstone bosons. In order to expand around the ground state $\varphi_{\mathrm{G}} = (v, \mathbf{0})$ of the spontaneously broken theory, one rewrites the Lagrangian with the shifted fields $\varphi = (\phi + v, \boldsymbol{\pi})$ and the massless Goldstone bosons π^a, and the massive field ϕ become visible in the Lagrangian,

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi + \partial_\mu\pi^a\partial^\mu\pi^a\right) - \frac{1}{2}\left(3gv^2 - m^2\right)\phi^2\\ &\quad + vK\phi - gv\phi^3 - \frac{g}{4}\phi^4 - \frac{g}{4}(\pi^a\pi^a)^2 + \frac{1}{2}K\pi^a\pi^a\\ &\quad - gv\phi\pi^a\pi^a - \frac{g}{2}\phi^2\pi^a\pi^a + j^0\phi + j^a\pi^a\,,\\ K &= m^2 - gv^2\,.\end{aligned} \tag{5}$$

To every order of the calculation, one has to determine v such that the vacuum expectation value vanishes,

$$\langle 0|\phi(x)|0\rangle = 0\,.$$

To one loop, the parameters have to be renormalized in the following way:

$$\begin{aligned} &g = \mu^{4-d}g_{\mathrm{r}}[1-24g_{\mathrm{r}}\lambda]\,,\quad m^2 = m_{\mathrm{r}}^2[1-12g_{\mathrm{r}}\lambda]\,,\\ &\varphi = Z^{\frac{1}{2}}\varphi_{\mathrm{R}}\,,\quad Z = 1 + O\left(g_{\mathrm{r}}^2\right)\,,\\ &\lambda = -\frac{1}{32\pi^2}\left(\frac{1}{\epsilon} + \Gamma'(1) + \ln(4\pi) + 1\right)\,,\\ &d = 4 - 2\epsilon\,. \end{aligned} \tag{6}$$

For the vacuum expectation value v, one obtains

$$\begin{aligned} v &= v_0\left[1 - \frac{3g_{\mathrm{r}}}{16\pi^2}\ln\left(\frac{2m_{\mathrm{r}}^2}{\mu^2}\right) + O\left(g_{\mathrm{r}}^2\right)\right]\,,\\ v_0 &= \mu^{-\epsilon}\frac{m_{\mathrm{r}}}{\sqrt{g_{\mathrm{r}}}}\,. \end{aligned} \tag{7}$$

Note that the scale μ is different from the scale $\bar\mu$.

3.2 Correspondence of the linear sigma model to chiral perturbation theory

As shown in [2], the generating functionals of the linear sigma model (equipped with additional external fields) in the heavy mass limit and chiral perturbation agree at first nonleading order, provided the low-energy constants of chiral perturbation theory are pertinent functions of the parameters of the linear sigma model.

We stick to our example, the scalar two-point function, and identify the corresponding quantity in the linear sigma model. The external field χ^a – which couples to the quark condensate – finds its counterpart in the external scalar source j^a.[1] Therefore, the counterpart of $H(s)$ is the renormalized scalar two-point function

$$G_{\mathrm{R}}^{(2,\mathbf{0})}(s) = \mathrm{i}Z\int \mathrm{d}^4x e^{\mathrm{i}px}\langle 0|T\phi(x)\phi(0)|0\rangle\,,\quad s = p^2\,, \tag{8}$$

for small external momenta s.

[1] This identity only holds up to a finite renormalization factor, which is a polynomial in the renormalized coupling constant g_{r}. However, this factor does not affect the leading logarithms.

3.3 Leading logarithm to two loops

We calculate the leading logarithms to one and two loops in the quantity $G_{\mathrm{R}}^{(2,\mathbf{0})}(s)$. In the following, we only quote the result and relegate the description of the calculation and the individual loop contributions to Appendix B.

It is evident that $G_{\mathrm{R}}^{(2,\mathbf{0})}(s)$ for small external momenta has the structure

$$G_{\mathrm{R}}^{(2,\mathbf{0})}\left(s,g_{\mathrm{r}},m_{\mathrm{r}}^2,\mu\right)=\frac{1}{2m_{\mathrm{r}}^2}\Big[c^{(0)}\left(s,m_{\mathrm{r}}^2,\mu\right)+c^{(1)}\left(s,m_{\mathrm{r}}^2,\mu\right)g_{\mathrm{r}}+c^{(2)}\left(s,m_{\mathrm{r}}^2,\mu\right)g_{\mathrm{r}}^2+O\left(g_{\mathrm{r}}^3\right)\Big]. \tag{9}$$

We now decompose the coefficients $c^{(i)}$ and indicate all logarithms that are possible at the corresponding order in g_{r}:

$$\begin{aligned}
c^{(0)}&=a_{0,0}^{(0)},\\
c^{(1)}&=a_{1,0}^{(1)}L_s+a_{0,1}^{(1)}L_m+a_{0,0}^{(1)},\\
c^{(2)}&=a_{2,0}^{(2)}L_s^2+a_{1,1}^{(2)}L_sL_m+a_{0,2}^{(2)}L_m^2+a_{1,0}^{(2)}L_s\\
&\quad+a_{0,1}^{(2)}L_m+a_{0,0}^{(2)},\\
&\vdots\\
L_s&=\ln\left(-\frac{s}{\mu^2}\right),\quad L_m=\ln\left(\frac{2m_{\mathrm{r}}^2}{\mu^2}\right).
\end{aligned} \tag{10}$$

The coefficients $a_{l,m}^{(k)}$ are polynomials in s/m_{r}^2. The indices of a coefficient $X_{k,l}^{(N,t)}$ always have the same meaning in the following: the lower indices k and l indicate the power of the momentum and mass logarithms L_s and L_m, respectively. The upper indices N and (if present) t stand for the order of the coupling constant g_{r} and the power of s/m_{r}^2, respectively.

In general, the coefficient $c^{(k)}$ can be written as a double sum:

$$c^{(k)}=\sum_{n=0}^{k}\sum_{l=0}^{k-n}a_{l,k-n-l}^{(k)}L_s^lL_m^{k-n-l}. \tag{11}$$

The coefficients $a_{l,m}^{(k)}$ are given by

$$\begin{aligned}
a_{0,0}^{(0)}&=1+\frac{1}{2}\frac{s}{m_{\mathrm{r}}^2}+\cdots,\\
a_{0,0}^{(1)}&=-\frac{3}{8\pi^2}-\frac{21}{64\pi^2}\frac{s}{m_{\mathrm{r}}^2}+\cdots,\\
a_{1,0}^{(1)}&=-\frac{3}{16\pi^2}-\frac{3}{16\pi^2}\frac{s}{m_{\mathrm{r}}^2}+\cdots,\\
a_{0,1}^{(1)}&=-\frac{3}{16\pi^2}-\frac{3}{16\pi^2}\frac{s}{m_{\mathrm{r}}^2}+\cdots,\\
a_{2,0}^{(2)}&=\frac{3}{256\pi^4}\frac{s}{m_{\mathrm{r}}^2}+\cdots,\\
a_{1,1}^{(2)}&=-\frac{9}{128\pi^4}-\frac{3}{128\pi^4}\frac{s}{m_{\mathrm{r}}^2}+\cdots,\\
a_{1,0}^{(2)}&=\frac{3}{128\pi^4}+\frac{21}{256\pi^4}\frac{s}{m_{\mathrm{r}}^2}+\cdots.
\end{aligned} \tag{12}$$

As an independent check of our loop calculation, we worked out the discontinuity of $G_{\mathrm{R}}^{(2,\mathbf{0})}(s)$,

$$G_{\mathrm{R}}^{(2,\mathbf{0})}(s+\mathrm{i}\epsilon)-G_{\mathrm{R}}^{(2,\mathbf{0})}(s-\mathrm{i}\epsilon)=2\mathrm{i}\pi\rho(s) \tag{13}$$

and compare with the discontinuity obtained from the optical theorem. The two expressions agree at the order considered. We refer to Appendix C for further details.

4 Linear sigma model versus ChPT

The translation rules provided in [2] are

$$\begin{aligned}
B&=\frac{\sqrt{g_{\mathrm{r}}}}{2m_{\mathrm{r}}}\left(1+\frac{1}{16\pi^2}\left(3L_m-1\right)g_{\mathrm{r}}+O\left(g_{\mathrm{r}}^2\right)\right),\\
F^2&=\frac{m_{\mathrm{r}}^2}{g_{\mathrm{r}}}\left(1-\frac{1}{16\pi^2}(6L_m-1)g_{\mathrm{r}}+O\left(g_{\mathrm{r}}^2\right)\right).
\end{aligned} \tag{14}$$

Note that the coupling constant g_{r} differs from the one introduced in [2] by a term of order g_{r}^2. The higher-order corrections to the above relations do not affect the coefficients of the leading logarithms $a_{N,0}^{(N)}$.

Translating with the above rules the coefficient of the one- and two-loop leading logarithms of the scalar correlator in (2) leads to

$$\begin{aligned}
\frac{B^2\bar{P}_1}{16\pi^2}&=\frac{1}{2m_{\mathrm{r}}^2}\left(-\frac{3}{16\pi^2}g_{\mathrm{r}}+\frac{3}{128\pi^4}(1-3L_m)g_{\mathrm{r}}^2+O\left(g_{\mathrm{r}}^3\right)\right)\\
&=\frac{1}{2m_{\mathrm{r}}^2}\left(a_{1,0}^{(1,0)}g_{\mathrm{r}}+\left(a_{1,0}^{(2,0)}+a_{1,1}^{(2,0)}L_m\right)g_{\mathrm{r}}^2+O\left(g_{\mathrm{r}}^3\right)\right),\\
\frac{B^2\bar{P}_2}{16\pi^2}&=\frac{1}{2m_{\mathrm{r}}^2}\cdot\frac{3}{256\pi^4}\frac{s}{m_{\mathrm{r}}^2}g_{\mathrm{r}}^2+O\left(g_{\mathrm{r}}^3\right)\\
&=\frac{1}{2m_{\mathrm{r}}^2}a_{2,0}^{(2,1)}g_{\mathrm{r}}^2+O\left(g_{\mathrm{r}}^3\right).
\end{aligned} \tag{15}$$

It is seen that they agree at the order considered. We have checked that the coefficients $a_{0,0}^{(0,0)}$, $a_{0,0}^{(1,0)}$ and $a_{0,1}^{(1,0)}$ agree as well. Therefore the coefficients of the one- and two-loop leading logarithms of the linear sigma model are the same as the coefficients of the one- and two-loop leading logarithms in chiral perturbation theory in this correlator.

We take this result as strong evidence that the leading logarithms of both theories agree to all orders in perturbation theory. Further support for this conjecture is the fact that, as shown in [4], the leading logarithms in the scalar two-point function in ChPT are determined by the tree-level amplitude. Stated differently, we believe that the linear sigma model acts as a renormalizable effective field theory for the leading logarithms in ChPT.

In the remaining part of this article, we assume that our conjecture is correct, and we work out its consequences: summing leading logarithms in the linear sigma model amounts to summing leading logarithms of the pertinent quantities in ChPT.

5 Renormalization group analysis in the linear sigma model

In this section, we illustrate the summation of leading logarithms with renormalization group techniques in the symmetric as well as in the spontaneously broken phase and investigate the low-energy structure of the correlator $G_{\mathrm{R}}^{(2,\mathbf{0})}(s)$.

5.1 Symmetric phase

Here we consider mass logarithms in the perturbative expansion of the physical mass (i.e., the position of the pole in the two-point function) in the symmetric phase of the linear sigma model. In particular, we recall how the leading, next-to-leading, etc. logarithms can be explicitly summed up.
First we recall the RGE in the unbroken phase of the linear sigma model for renormalized, Fourier transformed Green functions in four space-time dimensions $G_{\mathrm{R}}^{(\mathbf{n})}(p_i; g_{\mathrm{r}}, m_{\mathrm{r}}^2, \mu)$

$$\left(\mathcal{D}+\sum_{k=1}^{4} n_k \gamma\right) G_{\mathrm{R}}^{(\mathbf{n})}(p_i)=0\,; \qquad \mathbf{n}=(n_1,n_2,n_3,n_4)\,, \tag{16}$$

where

$$\begin{aligned} \mathcal{D} &= \mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g_{\mathrm{r}}}-m_{\mathrm{r}}^2\gamma_m\frac{\partial}{\partial m_{\mathrm{r}}^2}\,,\\ \beta &= \mu\frac{\partial}{\partial\mu}g_{\mathrm{r}}=\sum_{k=2}^{\infty}\beta^{(k)}g_{\mathrm{r}}^k=\frac{3}{2\pi^2}g_{\mathrm{r}}^2+O\left(g_{\mathrm{r}}^3\right)\,,\\ \gamma_m &= -\frac{1}{m_{\mathrm{r}}^2}\mu\frac{\partial}{\partial\mu}m_{\mathrm{r}}^2=\sum_{k=1}^{\infty}\gamma_m^{(k)}g_{\mathrm{r}}^k=-\frac{3}{4\pi^2}g_{\mathrm{r}}+O\left(g_{\mathrm{r}}^2\right)\,,\\ \gamma &= \frac{1}{2}\beta\frac{\partial}{\partial g_{\mathrm{r}}}\log Z=O\left(g_{\mathrm{r}}^2\right)\,. \end{aligned} \tag{17}$$

In the perturbative expansion, the physical mass has the structure

$$m_{\mathrm{ph}}^2=m_{\mathrm{r}}^2\left(k^{(0)}+k^{(1)}g_{\mathrm{r}}+k^{(2)}g_{\mathrm{r}}^2+\cdots\right)\,, \tag{18}$$

where

$$\begin{aligned} k^{(n)} &= k_n^{(n)}L_{\phi^4}^n+k_{n-1}^{(n)}L_{\phi^4}^{n-1}+\cdots+k_0^{(n)}\,; \quad k^{(0)}=1\,,\\ L_{\phi^4} &= \ln\left(\frac{m_{\mathrm{r}}^2}{4\pi\mu^2}\right)\,. \end{aligned} \tag{19}$$

The leading logarithms are fully determined by the one-loop expressions $\beta^{(2)}$ and $\gamma_m^{(1)}$. The proof of this statement (following the lines of [9]) starts from the observation that the physical mass obeys the homogeneous RGE

$$\mathcal{D}m_{\mathrm{ph}}^2=0\,. \tag{20}$$

Collecting the coefficients proportional to $g_{\mathrm{r}}^N L_{\phi^4}^{N-1}$, which must vanish individually, we find the recursion relation

$$-2Nk_N^{(N)}+\left\{(N-1)\beta^{(2)}-\gamma_m^{(1)}\right\}k_{N-1}^{(N-1)}=0\,; \qquad N=1,2,\ldots \tag{21}$$

It is seen that the one-loop expressions for the β- and γ_m-functions suffice to determine the coefficients $k_N^{(N)}$. In order to sum the logarithms, we introduce the quantities

$$\begin{aligned} f_i(x) &= \sum_{n=0}^{\infty}k_n^{(i+n)}x^n\,; \quad x=g_{\mathrm{r}}L_{\phi^4}\,,\\ m_{\mathrm{ph}}^2 &= m_{\mathrm{r}}^2\sum_{i=0}^{\infty}f_i(x)g_{\mathrm{r}}^i\,. \end{aligned} \tag{22}$$

The f_i correspond to the sum of terms along the tilted lines in Fig. 1; in particular, f_0 (f_1) denotes the sum of the leading (next-to-leading) logarithms. From the recursion relation (21) it follows that f_0 satisfies the differential equation

$$\left\{\left(2-x\beta^{(2)}\right)\frac{\mathrm{d}}{\mathrm{d}x}+\gamma_m^{(1)}\right\}f_0(x)=0\,, \tag{23}$$

from which one has

$$f_0(x)=\left(1-\frac{\beta^{(2)}}{2}x\right)^{\frac{\gamma_m^{(1)}}{\beta^{(2)}}}=\left(1-\frac{3}{4\pi^2}x\right)^{-\frac{1}{2}}\,. \tag{24}$$

The next-to-leading logarithms can be summed up in an analogous fashion. It is easy to convince oneself that one needs a two-loop calculation of the β- and γ_m-functions in this case.

5.2 Spontaneously broken phase

The derivation of the renormalization group equations for the linear sigma model in the spontaneously broken phase goes through exactly like in the unbroken phase,

$$\left(\mathcal{D}+\left(k+\sum_{t=1}^{3}j_t\right)\gamma\right)G_{\mathrm{R}}^{(k,\mathbf{j})}(p_i)=0\,; \qquad \mathbf{j}=(j_1,j_2,j_3)\,. \tag{25}$$

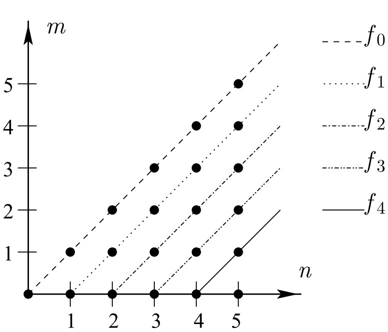

Fig. 1. Illustration of the structure of the physical mass in the symmetric phase of the linear sigma model. The quantity n represents the order in g_{r}, m stands for the exponent of the logarithm L_{ϕ^4}, the points represent the coefficients $k_m^{(n)}$, and the different dotted lines stand for the connection between them

Here we have denoted the renormalized Fourier transformed Green function with k (j) sigma (pion) fields by $G_{\mathrm{R}}^{(k,\mathbf{j})}$. As in the symmetric phase, it is straightforward to sum up the leading logarithms of quantities, which depend only on two scales. We illustrate this statement with the vacuum expectation value and the zero of the inverse sigma propagator.

5.2.1 Vacuum expectation value

The vacuum expectation value of the sigma field fulfills the inhomogeneous renormalization group equation

$$(\mathcal{D}+\gamma)v\left(g_{\mathrm{r}},m_{\mathrm{r}}^2,\mu\right)=0\,. \tag{26}$$

The perturbative series of v has the form

$$\begin{aligned} v&=\frac{m_{\mathrm{r}}}{\sqrt{g_{\mathrm{r}}}}\left(v^{(0)}+v^{(1)}g_{\mathrm{r}}+v^{(2)}g_{\mathrm{r}}^2+\cdots\right)\,,\\ v^{(n)}&=v_n^{(n)}L_m^n+v_{n-1}^{(n)}L_m^{n-1}+\cdots+v_0^{(n)}\,;\quad v^{(0)}=1\,. \end{aligned} \tag{27}$$

The recursion relation for the coefficients of the leading logarithms reads

$$2Nv_N^{(N)}+\left\{\beta^{(2)}\left(\frac{3}{2}-N\right)+\frac{1}{2}\gamma_m^{(1)}\right\}v_{N-1}^{(N-1)}=0\,. \tag{28}$$

Collecting again the leading logarithms in a function $h_0(x)$ leads to the differential equation

$$\left\{\frac{1}{2}\left(\beta^{(2)}+\gamma_m^{(1)}\right)+\left(2-\beta^{(2)}x\right)\frac{\mathrm{d}}{\mathrm{d}x}\right\}h_0(x)=0\,, \tag{29}$$

where $x=g_{\mathrm{r}}L_m$, with the solution

$$h_0(x)=\left(1-\frac{\beta^{(2)}}{2}x\right)^{\frac{\gamma_m^{(1)}+\beta^{(2)}}{2\beta^{(2)}}}=\left(1-\frac{3}{4\pi^2}x\right)^{\frac{1}{4}}\,. \tag{30}$$

5.2.2 The zero of the inverse sigma propagator

Next we investigate the zero $\hat{M}$ of the inverse sigma propagator. We denote by $\mathrm{Re}(\hat{M})$ its real part, and find

$$\begin{aligned} \mathrm{Re}(\hat{M})&=2m_{\mathrm{r}}^2\sum_{n=0}^{\infty}g_{\mathrm{r}}^n\sum_{i=0}^{n}b_i^{(n)}L_m^i\\ &=2m_{\mathrm{r}}^2\sum_{i=0}^{\infty}p_i(x)g_{\mathrm{r}}^i\,,\\ p_0(x)&=\left(1-\frac{\beta^{(2)}}{2}x\right)^{\frac{\gamma_m^{(1)}}{\beta^{(2)}}}=\left(1-\frac{3}{4\pi^2}x\right)^{-\frac{1}{2}}\,. \end{aligned} \tag{31}$$

Note that in the broken phase the functions $p_i(x)$ are the *same* as in the symmetric phase. Therefore, the coefficients of the mass logarithms in $\mathrm{Re}(\hat{M})$ and in the physical mass of the symmetric phase coincide up to a factor of 2.

6 Summing leading logarithms?

Here, we apply renormalization group techniques to the correlator $G_{\mathrm{R}}^{(2,\mathbf{0})}(s)$, written in the form (9), with an attempt to sum the leading logarithms $a_{N,0}^{(N)}L_s^N$. To start with, we insert the right hand side of (9) into the RGE (25). As the coefficients $a_{k,l}^{(n)}$ are analytic functions of $\frac{s}{m_{\mathrm{r}}^2}$ they can be represented by a power series,

$$a_{k,l}^{(n)}=\sum_{t=0}^{\infty}a_{k,l}^{(n,t)}\left(\frac{s}{m_{\mathrm{r}}^2}\right)^t=a_{k,l}^{(n,0)}+a_{k,l}^{(n,1)}\frac{s}{m_{\mathrm{r}}^2}+\cdots\,. \tag{32}$$

Analyticity demands the disappearance of the terms proportional to $L_s^iL_m^j$ individually and leads to the recursion relations for the leading momentum logarithms,

$$\begin{aligned} &-2Na_{N,0}^{(N,t)}-2a_{N-1,1}^{(N,t)}\\ &+\left((N-1)\beta^{(2)}+(1+t)\gamma_m^{(1)}\right)a_{N-1,0}^{(N-1,t)}=0\,. \end{aligned} \tag{33}$$

From this relation one concludes that in every order in s/m_{r}^2 such an equation exists. This is manifested by the index t. Furthermore, this recursion relation connects the coefficient of the leading logarithm at order g_{r}^N, $a_{N,0}^{(N,t)}$, with the coefficient of the leading logarithm at order g_{r}^{N-1}, $a_{N-1,0}^{(N-1,t)}$, and with the part of the coefficient of the next-to-leading logarithm at order g_{r}^N, which is proportional to one mass logarithm, $a_{N-1,1}^{(N,t)}$. In addition only the one-loop results of the β- and γ_m-function, $\beta^{(2)}$ and $\gamma_m^{(1)}$, appear in the recursion relation.

Comparing with the previous recursion relations for the physical mass and for the vacuum expectation value, one finds that in these relations only the coefficients of leading logarithms are involved. This fact allows the summation of the leading logarithms. In (33), however, the coefficients of the leading logarithms are no longer connected directly. This is illustrated in Fig. 2 by the dint of the solid line. First one could expect that there still exist recursion relations that allow for a direct connection between the coefficients of the leading logarithms. However, the dashed lines stand for recursion relations without leading logarithm coefficients and demonstrate that this idea is not successful. Therefore, the summation of the leading logarithms fails, since the troublesome coefficient $a_{N-1,1}^{(N,t)}$ is only determined by a N-loop calculation.

As seen above, only coefficients with power t in s/m_{r}^2 enter in the recursion relation. On the other hand the coefficient of the leading logarithm at order g_{r}^N is proportional to $(s/m_{\mathrm{r}}^2)^{N-1}$, hence this recursion relation cannot relate them.

Therefore, one cannot determine the leading logarithm at order g_{r}^N with the knowledge of the leading logarithm at lower order. For this reason the summation fails.

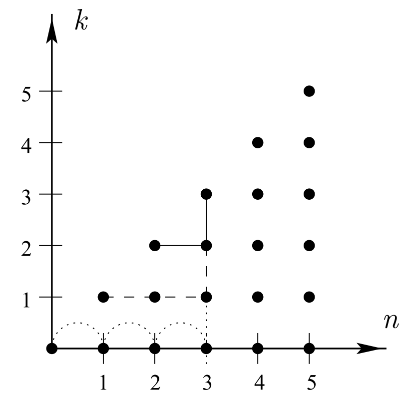

Fig. 2. Illustration of the connections between the coefficients of the scalar two-point function in the spontaneously broken phase of the linear sigma model at order g_{r}^3. The quantity n represents the order in g_{r}, and k stands for the exponent of the logarithm L_s. Every type of line indicates recursion relations containing the connected coefficients. There is one such picture for every order in $\frac{s}{m_{\mathrm{r}}^2}$

The situation becomes clear by introducing a new scale ρ and splitting up all mass and momentum logarithms as follows:

$$L_s = \ln\left(-\frac{s}{\rho^2}\right) + L_\mu\,, \quad L_m = \ln\left(\frac{2m_{\mathrm{r}}^2}{\rho^2}\right) + L_\mu\,, \\ L_\mu = \ln\left(\frac{\rho^2}{\mu^2}\right)\,. \tag{34}$$

Therefore, all terms of the form $g_{\mathrm{r}}^N L_s^k L_m^l$ with $k+l=N$ in the scalar two-point function (9) generate a logarithm L_μ^N. At a given order g_{r}^N, one is left with one μ-dependent logarithm with power N. The leading logarithms L_μ can be summed to all orders, as we show explicitly in Appendix D. It is now obvious that only all explicitly scale dependent logarithms L_μ can be summed with the help of the RGE.

In the case of the vacuum expectation value and the zero of the inverse propagator, the coefficients of the logarithms L_m and L_μ are the same, because there are only two scales involved. In the presence of three scales, this is no longer true, and a separation between the leading momentum logarithms L_s^N and other logarithms to the power N like $L_s^k L_m^l$ with $k+l=N$ is no longer possible with this technique.

Another way to have access to the recursion relation is the solution of the Callan–Symanzik equation, which provides a relation between n-point functions with momentum p_i and the scaled momentum p_i/ξ. But the recursion relations obtained in this way can be extracted from the ones worked out with the RGE. Therefore the Callan–Symanzik equation does not contain new information.

6.1 Linear sigma model with scale independent counterterms

In chiral perturbation theory, the leading logarithms are in principle always accessible with a one-loop calculation [5]. One might hope to transfer this method to the linear sigma model by using a formulation of the linear sigma model with scale independent counterterms, analogously to chiral perturbation theory. This formulation is discussed in [10, 11]. Studying the simplest case, we tried to calculate the one-loop leading momentum logarithm of the scalar two-point function with the help of the tree-level diagram containing the counterterm. One observes that only the sum of the coefficients of the leading momentum and the leading mass logarithm can be obtained in this manner. Therefore the statement is the same as with the recursion relations in the previous subsection.

7 Summary and conclusion

In this article, we investigate the structure of leading chiral logarithms in the correlator of two scalar quark currents, (1). In particular, we determine this correlator in the framework of the linear sigma model and compare the result with what is known from ChPT.

As a first step, we show that the leading logarithms agree in the two theories at order p^6 in the low-energy expansion (two-loop order). To the best of our knowledge, this is a new result and strongly suggests that the linear sigma model can be used as a *renormalizable* effective theory to calculate leading logarithms in $\mathrm{SU}(2)\times\mathrm{SU}(2)$ ChPT. The result also suggests that renormalization group techniques can be used to sum these terms. For this reason, we investigate the RG equation in the linear sigma model and use it to sum up leading mass singularities, e.g., in the vacuum expectation value of the sigma field.

Applying the same technique to the scalar two-point function $G_{\mathrm{R}}^{(2,\mathbf{0})}(s)$ – which is the analogue of the correlator $H(s)$ in (1) – allows one to work out recursion relations between the coefficients of the leading logarithms. We show that these recursion relations also contain subleading terms, which are not accessible by the renormalization group. As a result of this, given the leading logarithm at order g_{r}^N, the recursion relations do not allow one to calculate the leading logarithm at order g_{r}^{N+1}.

A summation of the explicit scale dependent leading logarithms is nonetheless always possible. However, if there are more than two scales involved, a separation between different types of leading logarithms like $\ln^N(-s/\mu^2)$ and $\ln^N(2m_{\mathrm{r}}^2/\mu^2)$, for example, is not possible. Therefore, an independent summation of the leading momentum logarithms fails, and it is only the sum of all coefficients of explicit scale dependent leading logarithms that is accessible. In the special case of only two scales (for example μ and m_{r}), the coefficients of the explicit scale dependent logarithms trivially agree with the coefficients of the leading mass logarithms.

To conclude, even if the linear sigma model represents an effective renormalizable theory for the leading logarithms of chiral perturbation theory, the summation of these leading logarithms by straightforward use of the renormalization group does not seem to be possible. However, using an alternative approach, a solution to the problem could still be feasible.

Acknowledgements. We would like to thank C. Greub and all the members of the institute for informative discussions. Furthermore, we are indebted to J. Gasser for a careful reading of the manuscript and many useful comments. This work was supported in part by the Swiss National Science Foundation, by RTN, BBW- Contract No. 01.0357, and EC-contract HPRN-CT2002-00311 (EURIDICE).

Appendix A: Triangle integrals

Most of the one- and two-loop integrals that are used in the loop calculations in Appendix B and C are provided in [12]. However, triangle integrals with one, two and three massless particles propagating in the loop are not considered there. This is the reason why we indicate the results of these vertex functions here:

$$\begin{aligned}
C^{(1)}(s,m^2) &= \int \frac{\mathrm{d}^d l}{\mathrm{i}(2\pi)^d} \frac{1}{m^2-(l+k_1)^2} \frac{1}{m^2-(l-k_2)^2} \frac{1}{-l^2} \\
&= \frac{1}{16\pi^2 m^2}\left(1+\frac{\tau}{12}+O(\tau^2)\right), \\
C^{(2)}(s,m^2) &= \int \frac{\mathrm{d}^d l}{\mathrm{i}(2\pi)^d} \frac{1}{m^2-l^2} \frac{1}{-(l+k_1)^2} \frac{1}{-(l-k_2)^2} \\
&= -\frac{1}{16\pi^2 m^2 \tau}\left(\mathrm{Li}_2(-\tau)+\ln(1+\tau)\ln(-\tau)\right), \\
\tau &= \frac{s}{m^2}, \quad s=(k_1+k_2)^2, \qquad \text{(A.1)}
\end{aligned}$$

and

$$\begin{aligned}
C^{(3)}(s,\mu) &= \mu^{4-d}\int \frac{\mathrm{d}^d l}{\mathrm{i}(2\pi)^d} \frac{1}{(l^2)^2} \frac{1}{(l-p)^2} \\
&= \frac{1}{16\pi^2 s}\Bigg\{-\frac{1}{\epsilon}+\gamma_\mathrm{E}+\ln\left(-\frac{s}{4\pi\mu^2}\right) \\
&\quad +\epsilon\left[\frac{1}{2}\zeta(2)-\frac{1}{2}\left(\ln\left(-\frac{s}{4\pi\mu^2}\right)+\gamma_\mathrm{E}\right)^2\right]\Bigg\} \\
&\quad +O(\epsilon^2), \\
s &= p^2. \qquad \text{(A.2)}
\end{aligned}$$

Appendix B: Two-loop calculation

In the two-loop calculation, we are only interested in the momentum logarithms. It is therefore sufficient to consider diagrams that develop a branch point at $s=0$. The set of one- and two-loop selfenergy diagrams that contribute to the discontinuity at threshold are shown in Fig. 3. The analytical expressions of the two-loop integrals can be found in [12], and we adopt the conventions used in this reference. We expand these expressions around $s=0$ by keeping the momentum logarithms and expanding the remaining part in a Laurent series in s. Evaluating

$$G_\mathrm{R}^{(2,\mathbf{0})}(s) = \frac{1}{M_\sigma^2 - s - \Sigma(s)}, \qquad \text{(B.1)}$$

where M_σ is the bare mass of the heavy particle which appears in the spontaneously broken phase,

$$M_\sigma^2 = 2m_\mathrm{r}^2\left[1-30g_\mathrm{r}\lambda-\frac{9g_\mathrm{r}}{16\pi^2}\ln\left(\frac{M_\sigma^2}{\mu^2}\right)+O\left(g_\mathrm{r}^2\right)\right], \qquad \text{(B.2)}$$

yields the result of (12). In Table 1 we indicate the contribution of each diagram N_x to the factor of L_s^2 in $G_\mathrm{R}^{(2,\mathbf{0})}(s)$

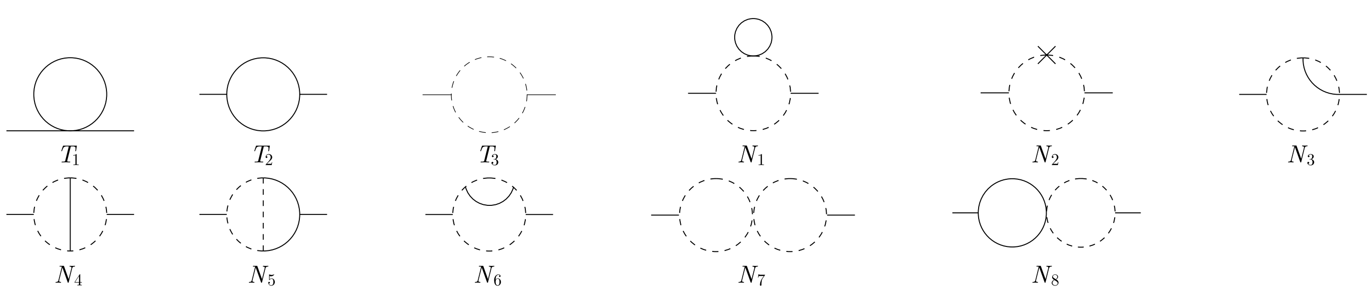

Fig. 3. Displayed are all the one-loop diagrams and the two-loop diagrams with a branch point at $s=0$. The *solid line* indicates a sigma particle and the *dashed line* a pion, respectively

Table 1. Contributions of the different diagrams to the factor of L_s^2. Every term has to be multiplied with $\frac{g_\mathrm{r}^2}{1024 m_\mathrm{r}^4 \pi^4}$

Diagram	Contribution	Diagram	Contribution	Diagram	Contribution
$T_1+T_2+T_3$	$90m_\mathrm{r}^2+99s$	N_3	$-24m_\mathrm{r}^2-24s$	N_6	$\frac{24m_\mathrm{r}^4}{s}+24m_\mathrm{r}^2+18s$
N_1	$\frac{12m_\mathrm{r}^4}{s}+12m_\mathrm{r}^2+9s$	N_4	$12m_\mathrm{r}^2+9s$	N_7	$-60m_\mathrm{r}^2-60s$
N_2	$-\frac{36m_\mathrm{r}^4}{s}-36m_\mathrm{r}^2-27s$	N_5	0	N_8	$-18m_\mathrm{r}^2-2s$

by inserting only $-\mathrm{i}N_x$ instead of the complete selfenergies Σ in (B.1). The terms proportional to $1/s$ that stem from the diagrams containing pion selfenergy parts as well a s the contributions without an s cancel each other.

Appendix C: Dispersive calculation

To calculate the discontinuity at two loops, the 1PI truncated diagrams shown in Fig. 4 and the one-loop diagrams indicated in Fig. 3 are required. We only need the analytical expressions of the diagrams for s small compared to the mass of the sigma particle. Performing the phase space integration

$$\rho(q^2) = (2\pi)^3 \frac{1}{2} \sum_{a,b} \int \mathrm{d}\mu(k_1)\, \mathrm{d}\mu(k_2) \delta^{(4)}(p - k_1 - k_2) \times \left| \langle 0|\phi(0)|\pi^a(k_1)\pi^b(k_2)\rangle \right|^2 , \tag{C.1}$$

where $\mathrm{d}\mu(k)$ is the Lorentz invariant measure,

$$\mathrm{d}\mu(k) = \frac{\mathrm{d}^3k}{(2\pi)^3 2k^0} , \tag{C.2}$$

leads to the discontinuity

$$\begin{aligned} \rho(s) = \frac{1}{16\pi^2} \Bigg[& \left(\frac{3}{2m_\mathrm{r}^2} + O(s) \right) g_\mathrm{r} + \Bigg(- \frac{3}{16\pi^2 m_\mathrm{r}^2} \\ & + \frac{9}{16\pi^2 m_\mathrm{r}^2} \ln\left(\frac{2m_\mathrm{r}^2}{\mu^2} \right) + O(s) \Bigg) g_\mathrm{r}^2 \\ & + \left(- \frac{3s}{16\pi^2 m_\mathrm{r}^4} + O(s^2) \right) \ln\left(\frac{s}{\mu^2} \right) g_\mathrm{r}^2 \\ & + O(g_\mathrm{r}^3) \Bigg] , \end{aligned} \tag{C.3}$$

which agrees exactly with the discontinuity calculated directly from our two-loop result.

Evaluating the phase space integration in d dimensions, we checked the discontinuities of the single two-loop diagram N_3, N_4, N_5, N_7 and N_8.

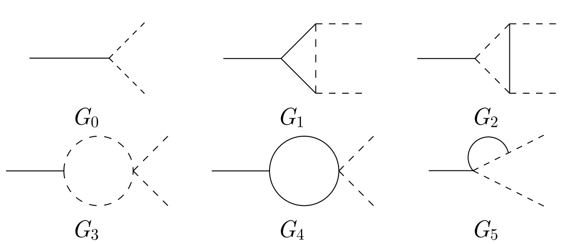

Fig. 4. Diagrams contributing to the matrix element $\langle 0|\phi(0)|\pi^a\pi^b\rangle$. The *solid line* denotes a sigma particle and the *dashed line* stands for a massless pion, respectively

Appendix D: Summation of scale dependent leading logarithms

Splitting up the mass and momentum logarithms as described in the text, it is possible to sum the logarithms L_μ to all orders. As described in Sect. 5, we collect all leading logarithmic terms that exhibit a factor of $\left(s/m_\mathrm{r}^2\right)^t$ in the function $f_t(x)$,

$$f(x) = \sum_{t=0}^{\infty} \left(\frac{s}{m_\mathrm{r}^2} \right)^t f_t(x) , \quad f_t(x) = \sum_{k=0}^{\infty} d_k^{(k,t)} x^k , \quad x = g_\mathrm{r} L_\mu . \tag{D.1}$$

Solving the corresponding differential equation with the initial condition $f_t(0) = 1/2^t$, one obtains

$$f_t(x) = \frac{1}{2^t} \left(1 - \frac{3}{4\pi^2} x \right)^{\frac{1}{2}(1+t)} . \tag{D.2}$$

The series in s/m_r^2 can also be summed and yields

$$G_\mathrm{R}^{(2,\mathbf{0})}(s) = \frac{1}{2m_\mathrm{r}^2 \left(1 - \frac{3}{4\pi^2}x\right)^{-\frac{1}{2}} - s} + \cdots , \tag{D.3}$$

where the ellipsis denotes all the subleading terms. Choosing $\rho^2 = 2m_\mathrm{r}^2$ and calculating the real part of the zero of the modified inverse propagator (D.3) one recovers the result from the Sect. 5.2.2.

To establish the connection to the recursion relations derived in Sect. 5, we express the coefficients d by means of the coefficients a,

$$d_N^{(N,t)} = a_{N,0}^{(N,t)} + a_{N-1,1}^{(N,t)} + \cdots + a_{0,N}^{(N,t)} . \tag{D.4}$$

Therefore, the function $f(x)$ includes the coefficients of the leading momentum logarithms. However, as we have seen in Sect. 5, the RGE do not allow to sum them separately.

References

1. S. Weinberg, Physica A **96**, 327 (1979)
2. J. Gasser, H. Leutwyler, Ann. Phys. **158**, 142 (1984)
3. J. Gasser, H. Leutwyler, Nucl. Phys. B **250**, 465 (1985)
4. M. Bissegger, A. Fuhrer, Phys. Lett. B **646**, 72 (2007) [arXiv:hep-ph/0612096]
5. M. Buchler, G. Colangelo, Eur. Phys. J. C **32**, 427 (2003) [arXiv:hep-ph/0309049]
6. D. Bettinelli, R. Ferrari, A. Quadri, arXiv:hep-th/0611063
7. B.W. Lee, Nucl. Phys. B **9**, 649 (1969)
8. K. Symanzik, Commun. Math. Phys. **16**, 48 (1970)
9. M.R. Ahmady et al., Phys. Rev. D **66**, 014010 (2002) [arXiv:hep-ph/0203183]
10. J. Bijnens, G. Colangelo, G. Ecker, J. Gasser, M.E. Sainio, Nucl. Phys. B **508**, 263 (1997)
11. J. Bijnens, G. Colangelo, G. Ecker, J. Gasser, M.E. Sainio, Nucl. Phys. B **517**, 639 (1998) [arXiv:hep-ph/9707291], Erratum
12. R. Scharf, J.B. Tausk, Nucl. Phys. B **412**, 523 (1994)